

DISTORTION FOR DIFFEOMORPHISMS OF SURFACES WITH BOUNDARY

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ABSTRACT. If G is a finitely generated group with generators $\{g_1, \dots, g_s\}$, we say an infinite-order element $f \in G$ is a distortion element of G provided that $\liminf_{n \rightarrow \infty} \frac{|f^n|}{n} = 0$, where $|f^n|$ is the word length of f^n with respect to the given generators. Let S be a compact orientable surface, possibly with boundary, and let $\text{Diff}(S)_0$ denote the identity component of the group of C^1 diffeomorphisms of S . Our main result is that if S has genus at least two, and f is a distortion element in some finitely generated subgroup of $\text{Diff}(S)_0$, then $\text{supp}(\mu) \subseteq \text{Fix}(f)$ for every f -invariant Borel probability measure μ . Under a small additional hypothesis the same holds in lower genus.

For μ a Borel probability measure on S , denote the group of C^1 diffeomorphisms that preserve μ by $\text{Diff}_\mu(S)$. Our main result implies that a large class of higher-rank lattices admit no homomorphisms to $\text{Diff}_\mu(S)$ with infinite image. These results generalize those of Franks and Handel [6] to surfaces with boundary.

1. INTRODUCTION

In this article, S is a compact orientable surface with boundary ∂S . We denote $\text{Int}(S) = S \setminus \partial S$. If $U \subseteq S$ is open (in the topology on S), we denote $\partial U = U \cap \partial S$, and $\text{Int}(U) = U \cap \text{Int}(S)$. μ will be a (not necessarily smooth) Borel probability measure on S . We denote the group of C^1 diffeomorphisms of S (preserving μ) by $\text{Diff}(S)$ (resp. $\text{Diff}_\mu(S)$), and we denote its identity component by $\text{Diff}(S)_0$ (resp. $\text{Diff}_\mu(S)_0$). The support of μ is denoted $\text{supp}(\mu)$. The set of fixed points of f is denoted $\text{Fix}(f)$, and the set of periodic points is denoted $\text{Per}(f)$.

Definition 1.1. If G is a finitely generated group, and we choose the generating set $\{g_1, \dots, g_s\}$, then $f \in G$ is said to be a *distortion element* of G if f has infinite order and

$$\liminf_{n \rightarrow \infty} \frac{|f^n|}{n} = 0,$$

where $|f^n|$ is the word length of f^n in the generators $\{g_1, \dots, g_s\}$.

See Gromov [8] for a good discussion with many examples.

Remark 1.2. We could have taken the \liminf to be a limit; because word length is subadditive, the limit must exist, and is called the *translation length* (see [7]).

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It is straightforward to see that the property of being a distortion element is independent of the finite generating set chosen. It is obvious that if $G' \supseteq G$ is also finitely generated, and f is distorted in G , then f is distorted in G' .

If G is not finitely generated, we say that $f \in G$ is distorted in G if it is distorted in some finitely generated subgroup of G .

One reason why distortion elements are interesting is that well-known groups have them. It is easy to check that the central elements of the three-dimensional Heisenberg group are distortion elements. Lubotzky, Mozes, and Raghunathan proved that irreducible nonuniform lattices in higher-rank Lie groups have distortion elements ([11]).

Franks and Handel [6] proved that for a closed surface S of genus at least two, if $f \in \text{Diff}(S)_0$ is a distortion element and μ is an f -invariant Borel probability measure on S , then $\text{supp}(\mu) \subseteq \text{Fix}(f)$. They also proved that the same holds for $S = S^2$ or T^2 , provided that f has at least three or one fixed points, respectively.

Our main theorem generalizes this result to the case of surfaces with boundary:

Theorem 1.3. *Let S be a compact surface, possibly with boundary. Let $f: S \rightarrow S$ be a C^1 diffeomorphism which is isotopic to the identity. Suppose f is distorted in $\text{Diff}(S)_0$. Suppose that the pair (S, f) satisfies the following property (\star) :*

Either $\text{genus}(S) \geq 2$, or

- *If $S = S^2$, f has at least three fixed points.*
- *If $S = D$ is a closed disk, f has at least one fixed point on ∂D or at least two fixed points in $\text{Int}(D)$.*
- *If $S = A$ is a closed annulus, f has at least one fixed point.*
- *If $S = T^2$, f has at least one fixed point.*

If μ is an f -invariant Borel probability measure, then $\text{supp}(\mu) \subseteq \text{Fix}(f)$. In particular, f has no non-fixed periodic points, since we can always put a finite invariant measure on a periodic orbit.

Remark 1.4. The theorem actually holds if $S = S^2$ and $f: S^2 \rightarrow S^2$ has only one fixed point. In that case, it is trivial, in the sense that it does not depend on f being a distortion element. Let x be the fixed point of f . If μ were an invariant probability measure not supported on $\{x\}$, we would have an action of f on $S^2 \setminus \{x\}$ with an invariant measure, hence recurrent points by the Poincaré recurrence theorem, and therefore fixed points in $S^2 \setminus \{x\}$ by the Brouwer plane translation theorem (see e.g. [3]), a contradiction. Therefore, an invariant probability measure will be supported on $\{x\}$.

Note that it is possible for a distortion element of $\text{Diff}(S^2)_0$ to have only one fixed point. Let f', g' , and $h': \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined in the following way: $g'(x, y) = (x + y, y)$, $h'(x, y) = (x, y + 1)$, and f' is the commutator $[g', h']$, so that $f'(x, y) = (x + 1, y)$. This is a faithful action of the three-dimensional (discrete) Heisenberg group, since g' and h' both commute with their commutator f' . Let ϕ be the diffeomorphism of \mathbb{R}^2 given by $\phi(x, y) = (\frac{x}{1+y^2}, y)$; we conjugate the above action of the Heisenberg group by ϕ , i.e. we take $f = \phi f' \phi^{-1}$, $g = \phi g' \phi^{-1}$, and $h = \phi h' \phi^{-1}$. The reason for conjugating is that g' is not differentiable at infinity, but f, g , and h are, with derivative the identity. So we can extend f, g , and h to be diffeomorphisms of $S^2 = \mathbb{R}^2 \cup \{\infty\}$; by a slight abuse of

notation we use the same names for these extensions. We have found a subgroup $\langle g, h \rangle$ of $\text{Diff}(S^2)_0$ isomorphic to the Heisenberg group, and the commutator f is a distortion element fixing only ∞ .

Question 1.5. Suppose $S = D$ is the closed disk, and f has only one fixed point, in $\text{Int}(D)$. Is it possible that there exists an f -invariant Borel probability measure μ , with support not contained in $\text{Fix}(f)$? In particular, is an irrational rotation of the disk distorted in $\text{Diff}(D)_0$?

The methods of Milson [12] on recurrent diffeomorphisms may be of use, but we do not immediately see how to apply them to surfaces with boundary.

Remark 1.6. Aside from the above cases, if (S, f) do not satisfy (\star) then there are counterexamples to the conclusion of the theorem. Calegari and Freedman [2] have shown that an irrational rotation of S^1 is distorted in $\text{Diff}^1(S^1)_0$. Foliating the closed annulus by circles, we get that an irrational rotation of the annulus A is distorted in $\text{Diff}^1(A)_0$. The same procedure works for the torus T^2 .

Calegari and Freedman proved in the same paper that irrational rotations of S^2 are distorted in $\text{Diff}^\infty(S^2)_0$, showing that the theorem does not hold for distortion elements in $\text{Diff}^1(S^2)_0$ with two fixed points.

Applications. Clearly, if S is a surface with boundary, and $f \in \text{Diff}(S)_0$ is a distortion element having enough fixed points (as given by Theorem 1.3), then f cannot preserve any measure with full support, such as area. Moreover, Theorems 1.5 and 1.6 and Corollaries 1.7 to 1.10 of [6] continue to remain valid for surfaces with boundary, with two caveats: sometimes additional hypotheses are needed in the genus 0 case, and there is a slight misprint in Corollary 1.9, which should say “ $\text{Diff}_\mu(S)_0$ ” instead of “ $\text{Diff}_\mu(S)$ ”.

For example, we have the following facts for a compact oriented surface S with nonempty boundary, equipped with a Borel probability measure μ :

- Let \mathcal{G} be an irreducible nonuniform lattice in a semisimple real Lie group of real rank at least two. Assume that the Lie group is connected, without compact factors, and with finite center. Suppose $S \neq D$ (the closed disk). Then every homomorphism $\phi: \mathcal{G} \rightarrow \text{Diff}_\mu(S)$ has finite image.
- Suppose that \mathcal{G} is a finitely generated, almost simple group that has a subgroup \mathcal{H} isomorphic to the three-dimensional Heisenberg group. Then any homomorphism $\phi: \mathcal{G} \rightarrow \text{Diff}_\mu(S)$ has finite image.
- Suppose that $\text{supp}(\mu) = S$. Suppose that $\mathcal{N} \subset \text{Diff}_\mu(S)_0$ is nilpotent. If $S \neq D$, then \mathcal{N} is abelian.

In verifying these claims, one uses the Thurston stability theorem ([13], Theorem 3), which remains valid for a surface with boundary. One also uses a result of Zimmer ([14], Theorem 3.14) that if \mathcal{G} is a finitely generated discrete group with Kazhdan property (T), and $\phi: \mathcal{G} \rightarrow \text{Diff}^1(S)$ has a finite orbit, then the image of ϕ is finite. This also works for surfaces with boundary; it relies on the fact that a Kazhdan group has no nontrivial homomorphism to \mathbb{R} and has only finite image homomorphisms to $\text{SL}(2, \mathbb{R})$, together with the Thurston stability theorem.

Question 1.7. How much of this remains true if we consider $\text{Homeo}(S)_0$ instead of $\text{Diff}(S)_0$?

2. NORMAL FORM

The first result we will need is that a C^1 diffeomorphism f of a compact surface with boundary is isotopic, relative to $\text{Fix}(f)$, to a *normal form* ϕ for f .

Definition 2.1. Let S be a compact surface with boundary. Let $f: S \rightarrow S$ be an orientation-preserving homeomorphism. Let $\phi: S \rightarrow S$ be another homeomorphism. We say that ϕ is isotopic to f relative to $\text{Fix}(f)$ if there is $F: S \times [0, 1] \rightarrow S$, a continuous family of homeomorphisms of S , such that $F(x, 0) = f(x)$ and $F(x, 1) = \phi(x)$ for all $x \in S$, and if $x \in \text{Fix}(f)$ then $F(x, t) = x$ for all t .

Following ideas of Handel [9] and Franks and Handel [5], we say that f *has a normal form relative to its fixed point set* if there is a finite set R of disjoint simple closed curves called *reducing curves* in $M = \text{Int}(S) \setminus \text{Fix}(f)$ and a homeomorphism ϕ isotopic to f relative to $\text{Fix}(f)$ such that:

- (1) ϕ permutes disjoint open annulus neighborhoods $A_j \subseteq M$ of the elements $\gamma_j \in R$.

Let $\{S_i\}$ be the components of $S \setminus \cup A_j$, let $X_i = \text{Fix}(f) \cap S_i$, let $M_i = S_i \setminus X_i$, and let r_i be the smallest positive integer such that $\phi^{r_i}(M_i) = M_i$. Note that $r_i = 1$ if $X_i \neq \emptyset$.

- (2) If X_i is infinite then $\phi|_{S_i} = \text{id}$.
- (3) If X_i is finite then M_i has negative Euler characteristic and $\phi^{r_i}|_{M_i}$ is either pseudo-Anosov or periodic. In the periodic case, $\phi^{r_i}|_{M_i}$ is an isometry of a hyperbolic structure on M_i .

The following says that Theorem 1.2 of [5] still holds for surfaces with boundary.

Theorem 2.2. *If f is a C^1 diffeomorphism of a compact surface with boundary, S , then f is isotopic relative to $\text{Fix}(f)$ to a homeomorphism ϕ which is a normal form for f .*

Lemma 2.3. *Let $A(f)$ denote the accumulation set of $\text{Fix}(f)$. It is possible to find a neighborhood V of $A(f)$ with finitely many components such that the following hold:*

- (1) $f|_V: V \rightarrow S$ is isotopic to the inclusion relative to $\text{Fix}(f) \cap V$.
- (2) If $x \in V \setminus \text{Fix}(f)$ then the path from $f(x)$ to x determined by the isotopy is contained in $S \setminus \text{Fix}(f)$.

Proof. We elaborate on the ideas of Lemma 4.1 of [9], and show that they remain valid with boundary. Choose a metric on S such that the boundary components are geodesics. Note that for any $x \in S$, there is a number d_x small enough so that for y within d_x of x , there is a unique shortest geodesic from x to y . Since S is compact, there is a number d which has this property for all $x \in S$. We may choose a neighborhood V of $A(f)$ small enough so that for all $x \in V$, $d(x, f(x)) < d$. Let $\gamma_x: [0, 1] \rightarrow S$ be the constant-speed parametrization of this geodesic from x to $f(x)$. In this way, we get a homotopy f_t between the inclusion $i: V \hookrightarrow S$ to $f|_V: V \rightarrow S$, where $f_t(x) = \gamma_x(t)$. We claim that, if V is small enough, this is actually an isotopy.

Let $a \in A(f)$. Then there is some $w \in T_a(S)$ fixed by Df_a . Therefore, since f is orientation-preserving, there cannot be $v \in T_a(S)$ such that $Df_a(v) = -kv$, with $k > 0$.

Note that this is also true if $a \in \partial S$, in which case $T_a(S)$ is the upper half plane. It is not hard to see that $(Df_t)_a(v) = t \cdot Df_a(v) + (1-t) \cdot v$. This implies that $(Df_t)_a(v) \neq 0$ for all $v \neq 0 \in T_a(S)$.

The rest follows without change from the proof of Handel [9], Lemma 4.1. Note that the components of V form an open cover of $A(f)$; since $A(f)$ is compact, we may find a finite subcover, so throwing out components if necessary we may assume that V has finitely many components. □

Proof of Theorem 2.2. The isotopy given by the above lemma may be extended to all of S , by applying the following theorem.

Theorem 2.4 (Morris Hirsch [10], Theorem 1.4, p. 180). *Let M be a manifold with boundary. Let $U \subset M$ be an open set and $A \subset U$ a compact set. Let $F: U \times I \rightarrow M$ be an isotopy of U , where $I = [0, 1]$. Let $\hat{F}: U \times I \rightarrow M \times I$ be defined by $\hat{F}(x, t) = (F(x, t), t)$. Suppose that $\hat{F}(U \times I) \subset M \times I$ is open. Then there is a diffeotopy of M having compact support, which agrees with F on a neighborhood of $A \times I$.*

In our situation, the open set will be V . Let the compact set A be the closure of a slightly smaller neighborhood of $A(f)$, such that $V \setminus A$ does not contain any fixed points of f . Let the manifold M be $S \setminus (\text{Fix}(f) \cap V^c)$; note that $\text{Fix}(f) \cap V^c$ is a finite set. The image $\hat{F}(U \times I)$ will be open, because if $x \in \partial S \cap V$ then under the isotopy given above the image of x will stay in ∂S (because the boundary components are geodesics), so a component of V touching a boundary component will continue to touch that boundary component under the isotopy. Therefore, by Theorem 2.4, there is a neighborhood $W \subseteq V$ of A and an isotopy F of M such that $F_0 = \text{id}$ and $F_1|_W = f|_W$.

Throwing out components of W if necessary, we may assume that each component of W contains an element of $A(f)$, so W consists of finitely many components. We may further assume that the components of W are smooth submanifolds; in particular, each has finitely many boundary components. Finally, we may assume that a component of W contains annular neighborhoods around all boundary circles in ∂S that it touches. This is for the following reason: if γ is a circle in ∂S , and C is a component of $S \setminus W$ intersecting γ , there will be at most finitely many fixed points of f in the interior of C ; none of them will accumulate on ∂C . Thus we may take a small neighborhood N of ∂C containing no fixed points of f , and there is no barrier to an isotopy to the identity on N , which can be extended to all of C by another application of Theorem 2.4. Note that adding annuli around boundary circles touched by components of W may have the effect of gluing components of W together.

The above argument holds with f^{-1} replacing f , so we can let F be an isotopy of M that goes from the identity to f^{-1} on W . Consider the isotopy $G_t = f \circ F_t$. $G_0 = f|_M$, and G_1 is a map which is the identity on W . This extends to an isotopy of S where we fill in the points of $\text{Fix}(f) \cap V^c$. This will be an isotopy relative to $\text{Fix}(f)$ from f to a map ϕ with $\phi|_W = \text{id}$.

The rest is quite similar to the proof of Theorem 1.2 of [5]. We look at the components of $S \setminus W$, which are compact surfaces with boundary. Note that if C is a component of $S \setminus W$, part of ∂C may lie in ∂S and part in $\text{Int}(S)$. Since part of ∂C may lie in ∂S , $\phi|_{\partial C}$

need not be the identity, which is a difference from [5]. But $\phi|_{\partial C}$ will be isotopic to the identity, and we may apply Theorem 1.3 of Hirsch [10] to C . The compact submanifold will be ∂C , and we will let F be an isotopy from $i: \partial C \hookrightarrow C$ to $\phi^{-1}|_{\partial C}$. By Hirsch this extends to an isotopy (which we also call) F of C . As above, we now take the isotopy $G_t = \phi \circ F_t$; close to the boundary this goes from ϕ to the identity. It extends to an isotopy of all of S by doing nothing outside C . Therefore, we may assume that $\phi|_{\partial C} = id$.

As in [5], if C is a disk with at most one element of $\text{Fix}(f)$, we can add it to W and after a further isotopy assume that $\phi|_W$ is still the identity. If $C = A$ is an annulus disjoint from $\text{Fix}(f)$, then after an isotopy either $\phi|_A$ is a Dehn twist or $\phi|_{W \cup A} = id$. To any other components of $S \setminus W$, we may puncture at the (finitely many) fixed points of f , and apply Thurston's decomposition theorem; we again call the result $\phi: S \rightarrow S$. Form R by taking any reducing curves coming from the Thurston decomposition of the components of $S \setminus W$, together with core curves of Dehn twist annuli, together with boundary curves of components of W not accounted for as core curves of Dehn twist annuli (except for components of ∂S).

After modifying ϕ on tubular neighborhoods of the reducing curves, we may assume that (1) is satisfied. Properties (2) and (3) are immediate from the construction. \square

Note: Since f is isotopic to the identity, ϕ will actually not permute the complementary components S_i given in the definition of the normal form; it will leave them invariant. This is a conclusion of Lemma 6.3 of [5] which remains valid when S has boundary.

3. AN IMPORTANT LEMMA

We need a couple of definitions, only slightly modified from Franks and Handel [6].

Definition 3.1. Let S be endowed with a Riemannian metric. A smooth curve γ has a well-defined length $\ell_S(\gamma)$. If τ is a smooth closed curve, define the *exponential growth rate* of τ with respect to f by

$$\text{egr}(f, \tau) = \liminf_{n \rightarrow \infty} \frac{\log(\ell_S(f^n(\tau)))}{n}.$$

Note that the exponential growth rate will be independent of the metric chosen on S .

Definition 3.2. Suppose that $f \in \text{Diff}(S)$, that M is a component of $S \setminus \partial S \setminus \text{Fix}(f)$ with negative Euler characteristic, that $h = f|_M$ is isotopic to the identity, and that β is an essential closed curve in M . If β is peripheral in M , assume that the end that it encloses is blown up to a boundary component. Choose a covering translation $T: \tilde{M} \rightarrow \tilde{M}$ whose axis Ax_T projects to a simple closed curve that is isotopic to β . Identify Ax_T with \mathbb{R} so that the action of T on Ax_T corresponds to a unit translation of \mathbb{R} . Let $\tilde{p}: \tilde{M} \rightarrow \mathbb{R}$ be a T -equivariant projection of \tilde{M} onto Ax_T (e.g. orthogonal projection) followed by the identification of Ax_T with \mathbb{R} . Let $\tilde{h}: \tilde{M} \rightarrow \tilde{M}$ be the identity lift of h . We say that $x \in M$ *linearly traces* β if there is a lift $\tilde{x} \in \tilde{M}$ such that

$$\liminf_{n \rightarrow \infty} \frac{|\tilde{p}(\tilde{h}(\tilde{x})) - \tilde{p}(\tilde{x})|}{n} > 0.$$

Note that any two T -equivariant projections of \tilde{M} onto Ax_T differ by a bounded amount, so Definition 3.2 is independent of the choice of projection.

Definition 3.3. Let U be a connected surface without boundary, with $H_1(U, \mathbb{Z}/2\mathbb{Z})$ finite. There exists a (essentially unique) surface U^* , without boundary, such that U is a topological subspace of U^* , and $U^* \setminus U$ is finite. The points in $U^* \setminus U$ are called “ends” of U .

Suppose U is contained in a connected surface without boundary, S , such that the closure of U in S is compact. We form a compactification \bar{U} of U associated with this embedding of U in S , which is a small modification of prime end compactification: instead of adding a single point to an end whose frontier consists of a single point, we perform a radial blow up. Given a C^1 diffeomorphism $f: S \rightarrow S$ such that $f(U) = U$, there is an extension of $f|_U$ to a homeomorphism $\bar{f}: \bar{U} \rightarrow \bar{U}$. We call this the *canonical extension* of $f|_U$ to \bar{U} . It respects the operation of composition: if g is another C^1 diffeomorphism of S sending U to U , then $\overline{f \circ g} = \bar{f} \circ \bar{g}$. (We get a similar composition-respecting extension to the prime end compactification; in that case, we may deal with homeomorphisms rather than diffeomorphisms.) For details, see [5], Lemma 5.1.

The following is the primary ingredient in proving the main result. It is a minor modification of Lemma 4.2 of [6]; parts (4) and (5) read slightly differently and part (3) is new.

Theorem 3.4. *Suppose that $f \in \text{Diff}_\mu(S)_0$ has infinite order, and $\text{supp}(\mu) \not\subseteq \text{Fix}(f)$. Suppose that (S, f) satisfies property (\star) . Then, after possibly replacing f with an iterate, at least one of the following holds:*

- (1) *There is a closed curve τ such that $\text{egr}(f, \tau) > 0$.*
- (2) *f is isotopic relative to $\text{Fix}(f)$ to a composition of nontrivial Dehn twists about a finite collection of nonperipheral, nonparallel, disjoint simple closed curves in $S \setminus \text{Fix}(f)$.*

For the following, f is isotopic to the identity relative to $\text{Fix}(f)$.

- (3) *There is a component C of ∂S such that $f|_C$ has irrational rotation number.*
- (4) *There is an f -invariant annular component U of $S \setminus \text{Fix}(f)$ such that the restriction of f to $\text{Int}(U) = U \setminus \partial S$ has canonical extension to $\bar{f}: \overline{\text{Int}(U)} \rightarrow \overline{\text{Int}(U)}$, and there is $x \in \text{Int}(U)$ such that the rotation number of x with respect to the lift of \bar{f} which fixes points on the boundary is nonzero.*
- (5) *There exists a component M of $S \setminus \partial S \setminus \text{Fix}(f)$ with negative Euler characteristic, a simple closed curve $\beta \subseteq M$ that is essential in M , and $x \in M$ that linearly traces β in M .*
- (6) *There exists $x \in S$ and a lift \tilde{f} such that $\text{Fix}(\tilde{f}) \neq \emptyset$ and such that the rotation vector $\rho(x, \tilde{f})$ is not zero. In this case, S has genus at least one, and \tilde{f} is the identity lift if $S \neq T^2$.*

Proof. We apply Theorem 2.2. If $\phi|_{S_i}$ is pseudo-Anosov for some Thurston component S_i , then there will be a closed curve $\tau \subseteq S_i$ with $\text{egr}(\phi, \tau) > 0$, hence $\text{egr}(f, \tau) > 0$, and we have (1).

If this does not hold, then letting ϕ_k be a normal form for f^k relative to $\text{Fix}(f^k)$ for each k , no ϕ_k will have a component on which it is pseudo-Anosov. Either there exists a k such that ϕ_k is a nontrivial composition of Dehn twists about simple closed curves, so after taking a finite power of f we have (2), or for every k , f^k is isotopic to the identity relative to $\text{Fix}(f^k)$.

We must show that, in this latter case, assuming (3) does not hold, at least one of (4), (5), or (6) must hold. By Brown and Kister [1], since f is orientation-preserving, every component of $S \setminus \text{Fix}(f)$ is invariant under f . Brown and Kister state their result for manifolds without boundary, but it still works here: if $f: M \rightarrow M$ is a homeomorphism of a manifold with boundary, we can simply apply their result to the restriction $f|_{\text{Int}(M)}$.

Since $\text{supp}(\mu) \not\subseteq \text{Fix}(f)$, there exists a component U of $S \setminus \text{Fix}(f)$ with $\mu(U) > 0$. By Poincaré recurrence, there are recurrent points in U . Suppose U is homeomorphic to a closed disk, possibly with part of its boundary removed. A priori this is possible, because a component of $S \setminus \text{Fix}(f)$ could contain parts of ∂S . $\partial U (= U \cap \partial S)$ cannot be a whole circle, or else f would be a diffeomorphism of the closed disk with no fixed points. But if $x \in \partial U$ then there is a whole (open) interval I in ∂U containing x , since $U \subseteq S$ is open. Because the boundary points of I in ∂S are fixed, $f(I) = I$, and all points in I must be sent in one direction, so they are not recurrent. Therefore, there must be a recurrent point in the interior of U . But by the Brouwer plane translation theorem this implies f has a fixed point in the interior of U , a contradiction.

Now suppose U is an annulus. U may be homeomorphic to an open annulus, a closed annulus, or something intermediate. The possibilities are as follows:

- (i) Both ends of U are open.
- (ii) Exactly one end of U is open.
 - (a) The open end has frontier consisting of a single point
 - (b) The open end has frontier consisting of more than one point
- (iii) Neither end of U is open.

We claim that $\bar{f}: \overline{\text{Int}(U)} \rightarrow \overline{\text{Int}(U)}$ has a fixed point on one of the two boundary circles of $\overline{\text{Int}(U)}$. For case (i) this is proved by Franks and Handel in their proof of Lemma 4.2 of [6]. Case (ii)(b) is identical; we have an open end whose frontier consists of more than a single point, so by prime end theory the circle of prime ends corresponding to that end is fixed pointwise.

In case (ii)(a), notice that the non-open end of U cannot be completely closed. If it were, it would consist of a single boundary component C of ∂S , and C would be peripheral in $S \setminus \text{Fix}(f)$. In this case, S is a closed disk, and f has a single interior fixed point, which is ruled out by the assumption that (S, f) satisfies property (\star) . Similarly, in case (iii), the ends cannot both be closed, because then S would be forced to be a closed annulus and f has no fixed points, which also violates the hypotheses of the theorem.

Suppose an end e of U is partly open and partly closed (the closed parts are arcs contained in ∂S). We form the prime end compactification $\overline{\text{Int}(U)}$, and we have $\bar{f}: \overline{\text{Int}(U)} \rightarrow \overline{\text{Int}(U)}$. Maximal arcs (connected components) in $U \cap \partial S$ have exactly corresponding arcs in $\partial \overline{\text{Int}(U)}$, and the dynamics of f and \bar{f} on these arcs are conjugate.

Let the arc α be a connected component of $U \cap \partial S$, with endpoints x_1 and x_2 . Without loss of generality, for any $x \in \alpha$, $f^n(x) \rightarrow x_1$ and $f^{-n}(x) \rightarrow x_2$ as $n \rightarrow \infty$. By the conjugacy, if $\bar{\alpha}$ denotes the corresponding arc in $\partial \overline{\text{Int}(U)}$, the endpoints of $\bar{\alpha}$ are fixed by \bar{f} . Thus, in all cases, we have established the claim.

By the reasoning of the above paragraph, any points in ∂U are wandering. Thus no measure can be supported on them, so $\mu(\text{Int}(U)) > 0$. Therefore, by Lemma 2.2 of [6], for every lift \tilde{f} of \bar{f} to the universal cover of $\overline{\text{Int}(U)}$ there is a point $x \in \text{Int}(U)$ such that $\rho(x, \tilde{f}) \neq 0$. In particular, this is true for a lift of \bar{f} fixing points on the boundary of the universal cover of $\overline{\text{Int}(U)}$. This implies that (4) holds.

We are finished exploring the possibilities if U is an annulus. Therefore, we may assume that U has negative Euler characteristic. We claim that $\text{Per}(f|_U) = \text{Fix}(f|_U)$. Since (3) does not hold, after taking a finite power if necessary we may assume that f has a fixed point, and no non-fixed periodic points, on each component of ∂S . There cannot be a non-fixed periodic point in the interior of U , as we can see by applying Lemma 3.7 of [4] to the restriction of f to the interior of U .

Since f has fixed points on each component of ∂S , any boundary arcs in U consist of wandering points and must have measure 0, so $\text{Int}(U)$ has positive measure. Analogously to [6], we apply the proof of Theorem 1.1 of [5] to $M := \text{Int}(U)$, as follows.

We use the notation and terminology of [5]. If there exists a simple closed geodesic γ such that, for a positive measure set $P \subseteq M$, for any $x \in P$ we have $\gamma(x) = \gamma$, then (5) holds. If there is an isolated puncture c in M such that for all x in a positive measure set $P \subseteq M$ we have that $\mathcal{O}(x)$ rotates about c , then again (5) holds.

Suppose there exists a simple closed geodesic $\alpha \subseteq M$ such that for a positive measure set $P \subseteq M$, for all $x \in P$, x is birecurrent and $\gamma(x)$ crosses α , and for all x such that $\gamma(x)$ crosses α the crossing is always in the same direction (so α is a “partial cross section”). In this case, note that S must have genus at least one, since by [5], Lemma 11.6(2), for all $x \in P$, $\gamma(x)$ is birecurrent, but if S had genus zero α would separate it into two components, and since $\gamma(x)$ is only allowed to cross α in one direction, after crossing it would have to stay in one component forever. In this case, the reasoning of Theorem 1.1 implies that $\rho_\mu(f_t) \wedge [\alpha] \neq 0$, so $\rho_\mu(f_t) \neq 0$, where f_t is an isotopy of f to the identity relative to $\text{Fix}(f)$, implying (6). \square

4. PROOF OF MAIN RESULT

We will need the following definitions, slightly modified from Franks and Handel [6].

Definition 4.1. We define *linear displacement* as follows. Let S be a surface of non-positive Euler characteristic. If S is closed, we follow [6]. Suppose S has nonempty boundary.

Let d be a metric on S , and \tilde{d} lift of d to the universal cover \tilde{S} . We will say that f has *linear displacement* if either of the following conditions holds:

- $S \neq A$, \tilde{f} is the identity lift, and there exists $\tilde{x} \in \tilde{S}$ such that

$$\liminf_{n \rightarrow \infty} \frac{\tilde{d}(\tilde{f}^n(\tilde{x}), \tilde{x})}{n} > 0.$$

- $S = A$, and there exists a lift \tilde{f} and $x_1, x_2 \in \tilde{A} = \mathbb{R} \times [0, 1]$ such that

$$\liminf_{n \rightarrow \infty} \frac{\tilde{d}(\tilde{f}^n(\tilde{x}_1), \tilde{f}^n(\tilde{x}_2))}{n} > 0.$$

Definition 4.2. We define *spread* in the following way. Suppose S is a compact surface (possibly) with boundary. Let γ be an embedded smooth path. Suppose that one of the following holds: the endpoints of γ lie in distinct components of ∂S ; one endpoint of γ is in ∂S and the other lies in $\text{Fix}(f)$; or the endpoints of γ are distinct elements of $\text{Fix}(f)$. Let β be a simple closed curve lying in S that crosses γ exactly once. Let A be the endpoint set of γ . If an endpoint of γ lies in $\text{Int}(S)$, remove it and blow up the puncture to a boundary circle; if it lies in ∂S , do nothing. Call the resulting surface M . We can think of γ and β as curves in M . Suppose for the moment that S has genus at least one; then M has negative Euler characteristic, and we choose a hyperbolic structure on M . Choose nondisjoint lifts $\tilde{\beta}, \tilde{\gamma} \subseteq \tilde{M}$, and let $T: \tilde{M} \rightarrow \tilde{M}$ be the covering translation corresponding to $\tilde{\beta}$. Denote $T^i(\tilde{\gamma})$ by $\tilde{\gamma}_i$. Each $\tilde{\gamma}_i$ is an embedded path in \tilde{M} . Moreover, $\tilde{\gamma}_i$ separates $\tilde{\gamma}_{i-1}$ from $\tilde{\gamma}_{i+1}$.

An embedded smooth path $\alpha \subseteq S$ whose interior is disjoint from $A \cap \text{Int}(S)$ can be thought of as a path in M . For each lift $\tilde{\alpha} \subseteq \tilde{M}$, there exist integers $a < b$ such that $\tilde{\alpha} \cap \tilde{\gamma}_i \neq \emptyset$ if and only if $a < i < b$. Define

$$\tilde{L}_{\tilde{\beta}, \tilde{\gamma}}(\tilde{\alpha}) = \max\{0, b - a - 2\}$$

and

$$L_{\beta, \gamma}(\alpha) = \max\{\tilde{L}_{\tilde{\beta}, \tilde{\gamma}}(\tilde{\alpha})\}$$

as $\tilde{\alpha}$ varies over all lifts of α .

Now suppose that the Euler characteristic of M is not negative, so it is 0. Thus M is a closed annulus. In this case, \tilde{M} is identified with $\mathbb{R} \times [0, 1]$, where $T(x, y) = (x + 1, y)$, and $\tilde{\gamma}$ goes from one boundary component to the other. With these modifications, $L_{\beta, \gamma}(\alpha)$ is defined as in the previous case.

There is an equivalent definition of $L_{\beta, \gamma}(\alpha)$ which does not involve covers. Namely, $L_{\beta, \gamma}(\alpha)$ is the maximum value k for which there exist subarcs $\gamma_0 \subseteq \gamma$ and $\alpha_0 \subseteq \alpha$ such that $\gamma_0 \alpha_0$ is a closed path that is freely homotopic to β^k relative to $A \cap \text{Int}(S)$. We allow the possibility that γ and α share one or both endpoints. The finiteness of $L_{\beta, \gamma}(\alpha)$ follows from the smoothness of the arcs α and γ .

Define the *spread* of α with respect to f, β , and γ to be

$$\sigma_{f, \beta, \gamma}(\alpha) = \liminf_{n \rightarrow \infty} \frac{L_{\beta, \gamma}(f^n \circ \alpha)}{n}.$$

Lemma 4.3. Suppose that $f \in \text{Diff}_\mu(S)_0$ has infinite order and that $\text{supp}(\mu) \not\subseteq \text{Fix}(f)$. Suppose that (S, f) satisfies property (\star) . Then, after possibly replacing f with an iterate, at least one of the following holds:

- (1) There is a closed curve τ such that $\text{egr}(f, \tau) > 0$.
- (2) f has linear displacement.
- (3) There is a k -fold cover S_k of S with $k = 1$ or 2 and a lift $f_k: S_k \rightarrow S_k$ of $f: S \rightarrow S$ which is isotopic to the identity, and α, β , and γ exist as in the definition of spread, such that $\sigma_{f_k, \beta, \gamma}(\alpha) > 0$.

(4) *There is a component C of ∂S such that $f|_C$ has irrational rotation number.*

Proof. We closely follow the proof of Corollary 5.5 of [6]. We may assume that $\partial S \neq \emptyset$, since the result is proved in [6] for closed surfaces. It's obvious that Theorem 3.4(1) implies Lemma 4.3(1) and Theorem 3.4(3) implies Lemma 4.3(4). If S is a surface (of genus at least one) with nonempty boundary and Theorem 3.4(6) holds, some point has nonzero rotation vector with respect to the identity lift, and Lemma 4.3(2) follows by definition of linear displacement.

Suppose Theorem 3.4(4) holds. As in [6], Corollary 5.5, we let β be the core curve of U . We let γ be an arc with interior in U which extends to an arc $\bar{\gamma}$ in \bar{U} such that the endpoints of γ in S are in $\text{Fix}(f)$; this is possible even if one end of U consists entirely of a component of ∂S since that component will have a fixed point of f . We let α be an arc with interior in U that has x as one endpoint (where x is a point with positive rotation number, as given in Theorem 3.4) and extends to an arc $\bar{\alpha}$ with endpoint in $\partial \bar{U}$ a fixed point of \bar{f} . Then $\sigma_{f,\beta,\gamma}(\alpha) > 0$, implying Lemma 4.3(3) with $k = 1$.

Suppose Theorem 3.4(5) holds, so there is a hyperbolic component M of $S \setminus \partial S \setminus \text{Fix}(f)$, a point $x \in M$, and a simple closed curve $\beta \subseteq M$ such that x linearly traces β in M . As in [6], Corollary 5.5, we may pass to a twofold cover of S if necessary to assume that there exists a smooth curve γ with interior in M , crossing β exactly once, with endpoints either in $\text{Fix}(f) \cap \text{Int}(S)$ or in ∂S . Then we apply the reasoning of Lemma 5.3 of [6] without change to conclude that $\sigma_{f,\beta,\gamma}(\gamma) > 0$, so Lemma 4.3(3) holds with $k = 1$ or 2 .

Finally, suppose Theorem 3.4(2) holds: (Some power of) f is isotopic, relative to $\text{Fix}(f)$, to a composition of Dehn twists about a collection $R(f)$ of disjoint, nonparallel (in $S \setminus \text{Fix}(f)$), nonperipheral simple closed curves. Let $R_e(f) \subseteq R(f)$ be those curves that are essential in S .

Suppose $R_e \neq \emptyset$. Then S cannot be the closed disk. If S has genus 0, it must have at least 2 boundary components. If $S = A$ is the closed annulus, we can find fixed points — possibly on ∂A ; note that we are assuming f has fixed points on each boundary component — x_1 and x_2 and a path τ connecting them such that τ has exactly one intersection with R_e . Then no lift of f to the infinite strip \tilde{A} fixes the full preimage of x_1 and x_2 , so we have Lemma 4.3(2). If $S \neq A$, then we can make use of the identity lift $\tilde{f}: \tilde{S} \rightarrow \tilde{S}$. We can find a point $x \in \text{Fix}(f)$, a lift $\tilde{x} \in \tilde{S}$, and a ray $\tilde{\sigma}$ connecting \tilde{x} to a point in S_∞ such that $\tilde{\sigma}$ crosses exactly one element of the full preimage \tilde{R}_e of R_e . In this case, \tilde{x} is not fixed by the identity lift \tilde{f} , so again Lemma 4.3(2) holds.

Now suppose that $R_e = \emptyset$, so every element of $R(f)$ bounds a disk in S which contains at least two points in $\text{Fix}(f)$. After passing to a twofold cover if necessary, we can assume it is not the case that all elements of $\text{Fix}(f)$ lie in the same disk in $S \setminus R(f)$. Therefore, choose x_1 and x_2 lying in different disks. Exactly as in [6], we can find a curve γ connecting them such that $\sigma_{f,\beta,\gamma}(\gamma) > 0$, so Lemma 4.3(3) holds. \square

Lemma 4.4. *Let S and $f: S \rightarrow S$ be as in the statement of Theorem 1.3. If item (4) of Lemma 4.3 holds, then f is not a distortion element in $\text{Diff}(S)_0$.*

Proof. First suppose S has genus 0. The number of boundary components plus the number of fixed points of f is at least 3. After blowing up fixed points or collapsing

boundaries to points if necessary, we may assume that in addition to C there is exactly one other boundary, C' , so $S = A$ is a closed annulus. There will be at least one fixed point, call it p .

Let \tilde{A} be the universal cover of A . Identify the boundary components of \tilde{A} with \mathbb{R} in such a way that translation by 1 corresponds to making a full circle around C or C' . Choose a lift \tilde{p} of p .

If $g: A \rightarrow A$ is an orientation-preserving homeomorphism preserving the boundaries, then we may lift it to a homeomorphism of \tilde{A} (in fact, there are countably many lifts, parametrized by \mathbb{Z}). If $g(p) = p$, there is a canonical lift \tilde{g} such that $\tilde{g}(\tilde{p}) = \tilde{p}$. This lifting preserves the group structure: if g and h are two homeomorphisms fixing p , then $\widetilde{h \circ g} = \tilde{h} \circ \tilde{g}$. Let us define $\hat{g}: \mathbb{R} \rightarrow \mathbb{R}$ to be the restriction of \tilde{g} to \tilde{C} (under its identification with \mathbb{R}), so $\widehat{h \circ g} = \hat{h} \circ \hat{g}$.

Suppose f is distorted in $\text{Diff}(S)_0$; let $f_1, \dots, f_s \in \text{Diff}(S)_0$, such that $f \in \langle f_1, \dots, f_s \rangle$, and such that $\liminf_{n \rightarrow \infty} \frac{|f^n|}{n} = 0$, where $|\cdot|$ is the word length in the generators f_1, \dots, f_s .

There is a number M such that for all $x \in \mathbb{R}$ and $i = 1, \dots, s$, $|\hat{f}_i(x) - x| < M$. This implies that $\liminf_{n \rightarrow \infty} \frac{|\hat{f}^n(0) - 0|}{n} \leq \liminf_{n \rightarrow \infty} \frac{M|f^n|}{n} = 0$. But \hat{f} is a lift of an irrational rotation, so it has irrational translation number, a contradiction.

The idea similar if the genus of S is ≥ 1 . Construct S_C from S by collapsing all boundaries of S except C to points. Form S' from S_C by filling in the hole bounded by C with a disk. The universal cover \tilde{S}' is (homeomorphic to) an open disk. Let $\tilde{S}_C \subseteq \tilde{S}'$ be the preimage of S_C (not the universal cover of S_C).

Let $g: S \rightarrow S$ be a homeomorphism isotopic to the identity. This yields a homeomorphism \hat{g} of S_C . We can choose a lift \tilde{g} of \hat{g} to \tilde{S}_C that preserves a specified component \tilde{C} of the preimage of C , and since g is isotopic to the identity \tilde{g} will preserve all components of the preimage of C . If $h: S \rightarrow S$ is also isotopic to the identity, then $\widetilde{h \circ g} = \tilde{h} \circ \tilde{g}$, since $\widetilde{h \circ g}$ and $\tilde{h} \circ \tilde{g}$ differ by at most a covering translation that fixes preimages of C .

Finally, we can form an annulus A by collapsing all preimages of C except \tilde{C} to points. Let p be one of these points coming from collapsing preimages of C . Then \tilde{g} induces a homeomorphism $\hat{\tilde{g}}$ of A which fixes p . We are now in exactly the situation above, since all the operations that led from g to $\hat{\tilde{g}}$ preserved group structure. \square

Proof of Theorem 1.3. We must show that each of the four items of Lemma 4.3 will imply that f cannot be a distortion element. If item (1) holds, then non-distortion follows exactly as in Lemma 6.3 of [6]. If item (2) holds, then whether $S = A$ or $S \neq A$ non-distortion follows exactly according to the reasoning of Lemma 6.1 of [6]. If item (4) holds, then we have non-distortion by Lemma 4.4.

Suppose item (3), positive spread, holds. Then the reasoning of Lemmas 6.6 and 6.8 of [6] remains valid, and we can slightly modify the reasoning of Lemma 6.7 of [6]. Let γ be as in our definition of spread; in particular, its endpoints are either fixed by f or lie in boundary components of S .

If the endpoints x and y of γ are both in $\text{Fix}(f) \cap \text{Int}(S)$, then the reasoning of Lemma 6.7 goes through without change, but note that $g_{x', y'}$ is only defined when $x', y' \in \text{Int}(S)$

(there is no diffeomorphism of S moving x and y to x' and y' if one of $\{x', y'\}$ lies in ∂S). If $x \in \text{Fix}(f) \cap \text{Int}(S)$ and $y \in \partial S$, then we can do the same procedure, except we only look at x', y' such that y' lies on the same component of ∂S as y . In either case, Lemma 6.8 continues to hold, so f is not distorted.

If both x and y are in different components of ∂S , then it is easier. In that case, if f were a distortion element in some finitely generated subgroup of $\text{Diff}(S)_0$ — say the group generated by f_1, \dots, f_n — then the f_i will leave invariant the boundary components of S . The analogue of Lemma 6.7 holds for the f_i , namely, there exists $K(f_i)$ such that if α is a curve as in the definition of spread, then

$$L_{\beta, \gamma}(f_i(\alpha)) \leq L_{\beta, \gamma}(\alpha) + K(f_i).$$

This is true because there is a uniform bound on the distance points in the universal cover move under any lift $\tilde{f}_i: \tilde{S} \rightarrow \tilde{S}$ to the universal cover of S . Thus we again conclude that f is not distorted. \square

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